

SEMIDEFINITE REPRESENTATION FOR CONVEX HULLS OF REAL ALGEBRAIC CURVES

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ABSTRACT. We prove that the closed convex hull of any one-dimensional semi-algebraic subset of \mathbb{R}^n has a semidefinite representation, meaning that it can be written as a linear projection of the solution set of some linear matrix inequality. This is proved by an application of the moment relaxation method. Given a nonsingular affine real algebraic curve C and a compact semi-algebraic subset K of its \mathbb{R} -points, the preordering $\mathcal{P}(K)$ of all regular functions on C that are nonnegative on K is known to be finitely generated. We prove that $\mathcal{P}(K)$ is stable, which means that uniform degree bounds exist for representing elements of $\mathcal{P}(K)$. We also extend this last result to the case where K is only virtually compact. The main technical tool for the proof of stability is the archimedean local-global principle. As a consequence from our results we establish the Helton-Nie conjecture in dimension two: Every convex semi-algebraic subset of \mathbb{R}^2 has a semidefinite representation.

INTRODUCTION

Let $K \subseteq \mathbb{R}^n$ be a real algebraic or semi-algebraic set. The question of how to represent the convex hull $\text{conv}(K)$ of K has attracted growing attention in recent years. A good part of this interest originates from optimization theory, namely from the problem of optimizing a linear functional over K . One of the most promising approaches that have been discussed is to express $\text{conv}(K)$ (at least up to taking closures) as a linear projection of a spectrahedron, that is, of a set described by a linear matrix inequality. In other words, one would like to find symmetric real matrices M_i, N_j of some size (for $0 \leq i \leq n, 1 \leq j \leq k$ and some k) such that, writing

$$M(x, y) = M_0 + \sum_{i=1}^n x_i M_i + \sum_{j=1}^k y_j N_j,$$

the closure of $\text{conv}(K)$ coincides with the closure of the set

$$S = \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^k \ M(x, y) \succeq 0\}. \quad (1)$$

(Here $M \succeq 0$ means that the symmetric matrix M is positive semidefinite.) In view of the very efficient methods available in semidefinite programming, such a representation is perfectly well suited for optimizing linear functionals over K .

Another approach tries to understand the set $\text{conv}(K)$ via the dual algebraic variety of the Zariski closure of its boundary, see [21] and [22] for more details.

The question of characterizing the class of projected spectrahedra was raised by Nemirovski in his plenary address at the ICM in Madrid [11]. Any projected spectrahedron is clearly a convex semi-algebraic set. No other restriction is currently known. Helton and Nie have conjectured [5] that conversely every convex semi-algebraic set has a semidefinite representation, by which one means, can be written as a projected spectrahedron. A variety of sufficient conditions for the existence of a semidefinite representation have been proved, see e.g. [5], [6], [14], [8], [12], [4],

[3], [16], [15], [30], and see also chapters 6 and 7 of the forthcoming book [2] for an overview of the current state of knowledge. In this paper we prove the existence of a semidefinite representation for the closed convex hull of any one-dimensional semi-algebraic set. From this we derive a proof of the Helton-Nie conjecture in dimension two.

For the construction of semidefinite representations we use the moment relaxation method that was introduced by Lasserre and Parrilo ([8], [17], see also [9], [2]). Computing the convex hull of a set $K \subseteq \mathbb{R}^n$ (that we assume to be basic closed semi-algebraic) means to determine the linear moments of all probability measures on K for which these moments exist. By considering finite-dimensional relaxations of the K -moment problem, one obtains a nested hierarchy $K(1) \supseteq K(2) \supseteq \dots$ of explicit projected spectrahedra that contain K . Let us assume that K is compact. Then the sets $K(d)$ approximate $\text{conv}(K)$ arbitrarily closely. When K is a compact real algebraic set, the approximating convex sets $TH(d) = \overline{K(d)}$ have been studied in the literature under the name *theta bodies* of K (see [4] and [2], ch. 7). The approximation is known to become exact, that is, $K(d) = \text{conv}(K)$ holds for some number d , if and only if every linear polynomial that is nonnegative on K has a weighted sum of squares representation with uniform degree bounds on the summands. See Theorem 2.4 below for a rigorous formulation.

We consider a nonsingular affine algebraic curve C over \mathbb{R} and a compact semi-algebraic subset K of $C(\mathbb{R})$, the set of real points on C . We work in $\mathbb{R}[C]$, the affine coordinate ring of C . The saturated preordering $\mathcal{P}(K)$ consists of all elements of $\mathbb{R}[C]$ that are nonnegative on K . It is known [25] that $\mathcal{P}(K)$ is finitely generated. This means that there exist finitely many elements $1 = h_0, h_1, \dots, h_r \in \mathcal{P}(K)$ such that every $f \in \mathcal{P}(K)$ has a representation

$$f = \sum_{i=0}^r \sum_j p_{ij}^2 h_i \quad (2)$$

with $p_{ij} \in \mathbb{R}[C]$. Fixing C , K and the h_i , the main result of this paper (Corollary 4.4) says that there exist universal degree bounds for such representations. That is, every $f \in \mathcal{P}(K)$ has some representation (2) in which the degrees of the summands are bounded above by some number that depends only on $\deg(f)$. (We are using degrees here to simplify the exposition, and so we tacitly assume that C is considered with a fixed embedding in some affine space.) Technically, this result is expressed by saying that the preordering $\mathcal{P}(K)$ is stable. From this it follows that, for any morphism $\phi: C \rightarrow \mathbb{A}^n$ into affine space of any dimension, the relaxation process for the convex hull of $\phi(K)$ in \mathbb{R}^n becomes exact. In fact, this latter property is equivalent to stability of $\mathcal{P}(K)$.

Our method for proving stability of $\mathcal{P}(K)$ may be of interest in that we do not show the existence of degree bounds directly. Rather, we establish the following equivalent fact: For any real closed field R containing \mathbb{R} , the preordering generated by the h_i in $R[C] = \mathbb{R}[C] \otimes R$ is again saturated (Theorem 4.3). This fact, in turn, is proved by an application of the archimedean local-global principle [26], which allows us to work in local rings. Since the field R is non-archimedean, this seems impossible at first. We are getting around this problem by working in the ring $B[C] = \mathbb{R}[C] \otimes B$, rather than in $R[C]$, where B is the smallest convex subring of R that contains \mathbb{R} (a non-noetherian valuation ring). We believe that this way of applying the archimedean local-global principle is novel and somewhat unexpected.

In the case where C has genus one and $K = C(\mathbb{R})$ is the full real curve (assumed to be compact), our main result is known by [30]. In that paper, using arguments of Riemann-Roch type, we had been able to give degree bounds of quite explicit nature, resulting in bounds for the sizes of the derived exact semidefinite representations.

For all curves of higher genus (as well as for genus one and $K \neq C(\mathbb{R})$), our present results should be new. In contrast to the method from [30], the techniques developed in the present paper do unfortunately not seem to give any explicit degree bounds.

From stability of $\mathcal{P}(K)$, for compact semi-algebraic sets K on nonsingular curves, we deduce the existence of a semidefinite representation for the convex hull of any compact semi-algebraic set $S \subseteq \mathbb{R}^n$ with $\dim(S) \leq 1$ (Theorem 5.2), using the moment relaxation process. This case in turn implies the existence of such a representation for the closed convex hull of any semi-algebraic set S with $\dim(S) \leq 1$, not necessarily compact (Theorem 6.1). From this we derive the Helton-Nie conjecture in dimension two (Theorem 6.7).

On the other hand, we extend the stability result to certain noncompact cases. Namely, when C is a nonsingular affine curve and $K \subseteq C(\mathbb{R})$ is a closed semi-algebraic set that is merely virtually compact (see 7.1), the saturated preordering $\mathcal{P}(K)$ is still finitely generated and stable (Theorem 7.2). Again, this is proved by a reduction to the compact case.

The paper is organized as follows. In Section 2 we give a brief account of the relaxation method for constructing semidefinite representations of convex hulls, in the generality that is needed here. Section 3 contains auxiliary results for working in the ring $\mathbb{R}[C] \otimes B$. This ring plays a key role in the proof of stability of $\mathcal{P}(K)$ in the compact case (Section 4). The existence of semidefinite representations for compact convex hulls is deduced in Section 5, whereas the extension to closed convex hulls of arbitrary one-dimensional sets is contained in Section 6. Finally, Section 7 contains the proof of stability in the virtually compact case.

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1. NOTATIONS AND PRELIMINARIES

By an affine variety over a field k we mean a reduced affine k -scheme of finite type, that is, the Zariski spectrum $V = \text{Spec}(A)$ of a finitely generated reduced k -algebra A . As usual, we also write $A = k[V]$. If E is any k -algebra, then $V(E) = \text{Hom}_k(A, E)$ denotes the set of E -valued points of V .

For the notion of real spectrum we refer to [1], [19], [10] or [28]. The real spectrum of a ring A is denoted by $\text{Sper}(A)$. A convenient way to think of the points of $\text{Sper}(A)$ is that every such point is represented by a homomorphism $A \rightarrow R$ into some real closed field R . Two homomorphisms $A \rightarrow R_i$ ($i = 1, 2$) represent the same point of $\text{Sper}(A)$ if, and only if, they can be amalgamated over A , that is, if and only if there exist A -embeddings $R_i \rightarrow R$ ($i = 1, 2$) into a common real closed field R .

For the notions of preordering and quadratic module in a ring A , see one of [19], [10] or [28]. Recall that a preordering T in A is called *saturated* if it contains every element of A which is nonnegative in all points of $\text{Sper}(A)$ where all elements of T are nonnegative.

Let R be a real closed field, and let V be an affine R -variety. Given a semi-algebraic subset K of $V(R)$, we denote the associated constructible subset of $\text{Sper } R[V]$ by \tilde{K} , as usual. The saturated preordering associated with K is denoted $\mathcal{P}(K)$, that is,

$$\mathcal{P}(K) = \{f \in R[V] : f|_K \geq 0\}.$$

The convex hull of a set $S \subseteq \mathbb{R}^n$ is denoted $\text{conv}(S)$. The set of extremal points of a convex subset $K \subseteq \mathbb{R}^n$ is denoted $\text{Ex}(K)$.

2. THE RELAXATION METHOD

2.1. Let A be a finitely generated \mathbb{R} -algebra, and let M be a finitely generated quadratic module in A , say $M = \Sigma_A h_0 + \cdots + \Sigma_A h_r$ with $1 = h_0, h_1, \dots, h_r \in A$ and $\Sigma_A := \Sigma A^2$ (the cone of sums of squares in A). The quadratic module M is said to be *stable* (see [18], [27]) if, given any finite-dimensional linear subspace U of A , there exist finite-dimensional linear subspaces W_0, \dots, W_r of A such that

$$M \cap U \subseteq \Sigma_{W_0} h_0 + \cdots + \Sigma_{W_r} h_r,$$

where Σ_{W_i} denotes the set of sums of squares of elements of W_i . This property does not depend on the choice of the generators h_0, \dots, h_r of M . If A is a polynomial ring over \mathbb{R} , then stability of M means that there exists a map $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ such that for every $f \in M$ there exists a representation $f = \sum_{i,j} p_{ij}^2 h_i$ with polynomials p_{ij} and with $\deg(p_{ij}^2 h_i) \leq \varphi(\deg(f))$ for all i, j .

2.2. By a *semidefinite representation* of a (convex semi-algebraic) set $S \subseteq \mathbb{R}^n$, one means a representation of S in the form (1), with suitable symmetric real matrices M_i, N_j . Any set S having such a representation is called a *projected spectrahedron*.

We now recall the method of moment relaxation [8] for constructing semidefinite representations, in a generality adapted to our needs. For more background we refer to Chapter 11 of [9], and to Chapters 5 and 7 of the forthcoming book [2]. We only outline the basic principle of the construction, ignoring possible refinements.

2.3. Let A be a finitely generated reduced \mathbb{R} -algebra. We denote the associated affine \mathbb{R} -variety by $V = \text{Spec}(A)$, and we always equip the set $V(\mathbb{R}) = \text{Hom}(A, \mathbb{R})$ of real points of V with its natural Euclidean topology. Fix elements $1 = h_0, h_1, \dots, h_r \in A$, write $\Sigma_A := \Sigma A^2$ for the cone of some of squares in A , let

$$M = \Sigma_A h_0 + \cdots + \Sigma_A h_r$$

be the quadratic module in A generated by the h_i , and let

$$K = \{\xi \in V(\mathbb{R}) : h_1(\xi) \geq 0, \dots, h_r(\xi) \geq 0\}$$

be the associated basic closed semi-algebraic subset of $V(\mathbb{R})$. We assume that K is Zariski dense in V . Fix a finite-dimensional vector subspace $L \subseteq A$ containing 1, and let $1, x_1, \dots, x_n$ be a basis of L . We consider the morphism $\phi = \phi_L = (x_1, \dots, x_n)$ from V to affine n -space determined by L , and the induced map $\phi: V(\mathbb{R}) \rightarrow \mathbb{R}^n$.

Fix a tuple $W = (W_0, \dots, W_r)$ of finite-dimensional linear subspaces of A , and consider the linear subspace

$$U := W_0 W_0 + h_1 W_1 W_1 + \cdots + h_r W_r W_r$$

of A . We assume that L is contained in U , and we denote by $\rho: U' \rightarrow L'$ the restriction map between the dual linear spaces. By U'_1 (resp. L'_1) we denote the set of all linear forms λ in U' (resp. in L') with $\lambda(1) = 1$. We identify \mathbb{R}^n with L'_1 via the map

$$L'_1 \xrightarrow{\sim} \mathbb{R}^n, \quad \lambda \mapsto (\lambda(x_1), \dots, \lambda(x_n)).$$

For $i = 0, \dots, r$ let $\Sigma_{W_i} \subseteq W_i W_i$ denote the cone of sums of squares of elements of W_i . The set

$$M_W := \Sigma_{W_0} + h_1 \Sigma_{W_1} + \cdots + h_r \Sigma_{W_r}$$

is contained in $M \cap U$ and is a convex semi-algebraic cone in U . Since K is Zariski dense in V , we have $M \cap (-M) = \{0\}$. This implies that M_W is closed in U ([18] Prop. 2.6). Let $M_W^* \subseteq U'$ be the dual cone of M_W . Then M_W^* can be defined by a

(homogeneous) linear matrix inequality, that is, M_W^* is a spectrahedral cone in U' . The subset $M_W^* \cap U'_1$ of M_W^* is therefore a spectrahedron as well. Its image set

$$K_W := \rho(M_W^* \cap U'_1) = L'_1 \cap \rho(M_W^*) \subseteq \mathbb{R}^n$$

under the restriction map $\rho: U'_1 \rightarrow L'_1 = \mathbb{R}^n$ is therefore a projected spectrahedron in \mathbb{R}^n . For every $\xi \in K$, the cone M_W^* obviously contains the evaluation map at ξ (restricted to U). Therefore K_W contains the set $\phi(K)$, and hence we also have $\text{conv}(\phi(K)) \subseteq K_W$. Increasing the subspaces W_0, \dots, W_r of A results in making K_W smaller. The main facts are summarized in the following theorem (c.f. [8] Theorem 2):

Theorem 2.4.

- (a) $\overline{K_W} = \{\eta \in \mathbb{R}^n : \forall f \in L \cap M_W \ f(\eta) \geq 0\}$;
- (b) the inclusion $\overline{\text{conv}(\phi(K))} \subseteq \overline{K_W}$ of closed convex sets is an equality if and only if $L \cap \mathcal{P}(K) \subseteq M_W$;
- (c) if M is archimedean then $\text{conv}(\phi(K)) = \bigcap_W K_W$, intersection over all systems $W = (W_0, \dots, W_r)$ of finite-dimensional subspaces of A .

If K is compact then $\text{conv}(\phi(K))$ is again compact by Carathéodory's lemma, and we get:

Corollary 2.5. *If K is compact, then $\text{conv}(\phi(K)) = K_W$ holds if and only if $L \cap \mathcal{P}(K) \subseteq M_W$.*

2.6. The moment relaxation is said to become *exact* when equality $\overline{\text{conv}(\phi(K))} = \overline{K_W}$ holds for some choice of finite-dimensional subspaces $W = (W_0, \dots, W_r)$. When K is compact, this is equivalent to $\text{conv}(\phi(K)) = K_W$.

If one is aiming at describing the convex hull of $\phi(K)$ in \mathbb{R}^n (approximately or exactly), note that there is a two-fold freedom of modifying the above construction. On the one hand, we may enlarge the subspaces W_0, \dots, W_r . We may as well enlarge the quadratic module M by adding (finitely many) more generators h_i from $\mathcal{P}(K)$. Both steps result in making the approximation tighter. When the saturated preordering $\mathcal{P}(K)$ is itself finitely generated, then choosing $M = \mathcal{P}(K)$ will give the best approximations for $\text{conv}(\phi(K))$.

When $K = V(\mathbb{R})$ is a real algebraic set, and when an embedding $V \subseteq \mathbb{A}^n$ has been fixed, the closed convex sets $\overline{K_W} \subseteq \mathbb{R}^n$ resulting from $M = \Sigma\mathbb{R}[V]^2$ approximate the closed convex hull $\overline{\text{conv}(V(\mathbb{R}))}$. They have been studied under the name *theta bodies* by Gouveia, Parrilo, Thomas and others (see [4] and [2], Chapter 7).

Varying the embedding ϕ , we see:

Corollary 2.7. *Let V be an affine \mathbb{R} -variety, let $K \subseteq V(\mathbb{R})$ be a basic closed set, Zariski dense in V , and assume that the saturated preordering $\mathcal{P}(K)$ in $\mathbb{R}[V]$ is finitely generated. Then the following two conditions are equivalent:*

- (i) *For any $n \in \mathbb{N}$ and any morphism $\phi: V \rightarrow \mathbb{A}^n$ of \mathbb{R} -varieties, the moment relaxation for the closed convex hull $\overline{\text{conv}(\phi(K))}$ becomes exact;*
- (ii) *the preordering $\mathcal{P}(K)$ in $\mathbb{R}[V]$ is stable.*

Proof. After fixing a finite description $\mathcal{P}(K) = \Sigma h_0 + \dots + \Sigma h_r$ (with $\Sigma = \Sigma\mathbb{R}[V]^2$), stability of $\mathcal{P}(K)$ means that for every finite-dimensional subspace $L \subseteq \mathbb{R}[V]$ there exists a tuple $W = (W_0, \dots, W_r)$ of finite-dimensional subspaces such that $L \cap \mathcal{P}(K) \subseteq M_W$. According to 2.4(b), this is equivalent to (i). \square

3. AUXILIARY RESULTS

Let C be a nonsingular curve over \mathbb{R} . Here we collect results that are needed for working in the base extension of C to a real closed valuation ring $B \supseteq \mathbb{R}$. The

situation has some resemblance with arithmetic surfaces. The main tool that will be needed in the next section is Proposition 3.13.

3.1. The following situation will be fixed. Let R be a real closed field containing \mathbb{R} , the field of real numbers, and let $B \subseteq R$ be the convex hull of \mathbb{R} in R . Then B is a valuation ring of R , and we denote by $v: R \rightarrow \Gamma \cup \{\infty\}$ the associated Krull valuation. The maximal ideal of B will be denoted by \mathfrak{m} . The residue field is $B/\mathfrak{m} = \mathbb{R}$.

Let A be a finitely generated \mathbb{R} -algebra, write $A_B = A \otimes B$ and $A_R = A \otimes R$ (with $\otimes := \otimes_{\mathbb{R}}$ always). Given $0 \neq f \in A_R$ we can write $f = \sum_{i=1}^r a_i \otimes b_i$ with $a_i \in A$ and $b_i \in R$ in such a way that a_1, \dots, a_r are linearly independent over \mathbb{R} . Putting

$$w(f) := \min\{v(b_i) : i = 1, \dots, r\}$$

and $w(0) := \infty$ gives a well-defined map $w: A_R \rightarrow \Gamma \cup \{\infty\}$ which extends the valuation v . For $f, g \in A_R$ we clearly have

$$w(f + g) \geq \min\{w(f), w(g)\}$$

and

$$w(fg) \geq w(f) + w(g),$$

and for $b \in R$ we have $w(bf) = w(f) + v(b)$.

The residue map $\pi: B \rightarrow B/\mathfrak{m} = \mathbb{R}$ will often be denoted $b \mapsto \bar{b}$. Accordingly we often denote the induced homomorphism $A_B \rightarrow A$ by $f \mapsto \bar{f}$. We have $A_B = \{f \in A_R : w(f) \geq 0\}$, and for $f \in A_B$ we have $\bar{f} = 0$ iff $w(f) > 0$.

Lemma 3.2. *Assume that the \mathbb{R} -algebra A is integral. Then $w(fg) = w(f) + w(g)$ holds for all $f, g \in A_R$, and so w extends to a valuation of the field of fractions of A_R .*

Clearly, the residue field of the valuation w of $\text{Quot}(A_R)$ is $\text{Quot}(A)$.

Proof. A integral implies that A_R is integral, too. We can write $f = af_0$ and $g = bg_0$, where $a, b \in R$ and $f_0, g_0 \in A_B$ satisfy $w(f_0) = w(g_0) = 0$. So we can assume $w(f) = w(g) = 0$, which means $\bar{f}, \bar{g} \neq 0$. Since A is a domain we have $\bar{f} \cdot \bar{g} \neq 0$, which implies $w(fg) = 0$. The lemma is proved. \square

3.3. Write $V = \text{Spec}(A)$ for the affine \mathbb{R} -scheme associated with A . We need to work with the real spectrum of $A_B = A \otimes B$. Given any point $\xi \in V(\mathbb{C}) = \text{Hom}_{\mathbb{R}}(A, \mathbb{C})$, we'll denote by M_ξ the kernel of the homomorphism $\xi \otimes \pi: A \otimes B \rightarrow \mathbb{C}$. So

$$M_\xi := \left\{ \sum_i a_i \otimes b_i \in A \otimes B : \sum_i a_i(\xi) \bar{b}_i = 0 \text{ in } \mathbb{C} \right\}.$$

Clearly, M_ξ is a maximal ideal of $A \otimes B$ whose residue field is the residue field of ξ . When ξ is real, i.e. $\xi \in V(\mathbb{R})$, there is a unique point in $\text{Sper}(A \otimes B)$ whose support is M_ξ . This point will be denoted α_ξ .

We fix a semi-algebraic subset K of $V(\mathbb{R})$ and, as usual, denote by \tilde{K} the constructible subset of $\text{Sper}(A)$ corresponding to K . The natural homomorphism $i: A \rightarrow A_B$ induces a continuous map $i^*: \text{Sper}(A_B) \rightarrow \text{Sper}(A)$ of the real spectra, and we write $X_K := (i^*)^{-1}(\tilde{K})$. So X_K is a constructible subset of $\text{Sper}(A_B)$, which is closed if K is closed in $V(\mathbb{R})$. By K_R we denote the semi-algebraic subset of $V(R)$ that is defined by the same inequalities as K . Considering $V(R)$ as a subset of $\text{Sper}(A_B)$ in the natural way, we have $K_R = V(R) \cap X_K$.

Proposition 3.4. *Assume that the semi-algebraic set K is compact. Then the closed points of X_K are precisely the points α_ξ , for $\xi \in K$.*

Proof. For $\xi \in K$ we have $\alpha_\xi \in X_K$ by construction, and this is a closed point of $\text{Sper}(A_B)$ since $\text{supp}(\alpha_\xi) = M_\xi$ is a maximal ideal of A_B . Conversely, let $\alpha \in X_K$ be a point that is represented by a homomorphism $\phi: A \otimes B \rightarrow S$ into a real closed field S . Let $C \subseteq S$ be the convex hull of \mathbb{R} in S , so we have $C/\mathfrak{m}_C = \mathbb{R}$. We claim that $\text{im}(\phi) \subseteq C$ holds. Indeed, let $a \in A$ and $b \in B$. Since K is compact there is $c \in \mathbb{R}$ with $|a| < c$ on K , and it follows that $|\phi(a \otimes 1)| < c$ in S . On the other hand, there is a real number $c' > 0$ such that $|b| < c'$ holds on $\text{Sper}(B)$, for example $c' = 1 + |\bar{b}|$. So we get $|\phi(a \otimes b)| < cc'$ in S , whence $\phi(a \otimes b) \in C$. We can therefore compose $\phi: A \otimes B \rightarrow C$ with the residue homomorphism $C \rightarrow \mathbb{R}$, resulting in a homomorphism $\psi: A \otimes B \rightarrow \mathbb{R}$. The point $\beta \in \text{Sper}(A \otimes B)$ represented by ψ is a specialization of α . On the other hand, $\beta = \alpha_\xi$ for some $\xi \in V(\mathbb{R})$, and the point ξ lies in K . \square

Lemma 3.5. *Let $K \subseteq V(\mathbb{R})$ be a semi-algebraic set, and let $f \in A_B$. Then f is nonnegative on the subset X_K of $\text{Sper}(A \otimes B)$ if, and only if, f is nonnegative on $K_R \subseteq V(R)$.*

Proof. The “only if” part is clear since $K_R \subseteq X_K$, see 3.3. Conversely assume $f \geq 0$ on K_R , and let $\alpha \in X_K$. There exist a convex subring W of R that contains B , a real closed field E containing $\kappa_W = W/\mathfrak{m}_W$ and a point $\xi \in K_E \subseteq V(E)$ such that α is represented by the homomorphism $A \otimes B \rightarrow E$, $a \otimes b \mapsto a(\xi)\bar{b}$.

There exists a homomorphic section of the residue map $W \rightarrow \kappa_W$. Let $s: \kappa_W \rightarrow R$ be the field embedding induced by such a section. After enlarging E if necessary, there exists a κ_W -embedding $\psi: R \hookrightarrow E$:

$$\begin{array}{ccc} \kappa_W & \xrightarrow{\quad} & E \\ s \downarrow & \nearrow \psi & \\ R & & \end{array}$$

The homomorphism $A \otimes R \rightarrow E$, $a \otimes b \mapsto a(\xi) \cdot \psi(b)$ represents a point $\beta \in \text{Sper}(A_R)$ that lies in $\widehat{K_R}$. Let $\beta_0 \in \text{Sper}(A_B)$ be the image of β under the inclusion $\text{Sper}(A_R) \subseteq \text{Sper}(A_B)$. Then β_0 lies in X_K and specializes to α . Since $f \geq 0$ on K_R by assumption, we have $f \geq 0$ in β resp. in β_0 , and therefore also $f \geq 0$ in α . \square

3.6. Now we specialize to the case where C is an integral affine curve over \mathbb{R} , and $A = \mathbb{R}[C]$ is the affine coordinate ring of C . We keep fixing the extension $\mathbb{R} \subseteq R$ of real closed fields and the convex hull B of \mathbb{R} in R , and we'll write $R[C] := A \otimes R$ and $B[C] := A \otimes B$. The following lemma is specific to the curves case.

Lemma 3.7. *Let C be an integral affine curve over \mathbb{R} , and let $K \subseteq C(\mathbb{R})$ be a compact semi-algebraic set. Let M be a maximal ideal of $\mathbb{R}[C] \otimes B$, and assume that there exists $\alpha \in X_K$ with $\text{supp}(\alpha) \subseteq M$ and with $\text{supp}(\alpha) \not\subseteq \mathbb{R}[C] \otimes \mathfrak{m}$. Then $M = M_\xi$ for some $\xi \in K$.*

Proof. Write $A = \mathbb{R}[C]$ as before. Let $P = \text{supp}(\alpha)$ and write $\mathfrak{p} = P \cap A$. From $P \not\subseteq A \otimes \mathfrak{m}$ it follows that $\mathfrak{p} \neq (0)$, and therefore \mathfrak{p} is a maximal ideal of A . Since P supports a point in X_K we see that \mathfrak{p} is the maximal ideal of A corresponding to some point $\xi \in K$. Since $P' := \mathfrak{p} \otimes B$ is contained in P , and since $(A \otimes B)/P' = B$, we conclude that necessarily $M = M_\xi$. \square

3.8. We keep fixing the extension $\mathbb{R} \subseteq R$ and the valuation ring B of R as before. We assume now that C is a nonsingular and geometrically connected affine algebraic curve over \mathbb{R} , and we consider the affine scheme $C \otimes_{\mathbb{R}} B = \text{Spec}(\mathbb{R}[C] \otimes B)$. This is a relative affine curve over $\text{Spec}(B)$. If B were a discrete valuation ring, the situation

would be a (very particular) instance of a relative curve over a Dedekind scheme, hence an arithmetic surface. However, B has divisible value group and therefore is not noetherian. Moreover, the Krull dimension of B can be arbitrarily large. Therefore we cannot directly rely on arguments that are well-known for arithmetic surfaces, or simply for noetherian rings. Still, the situation and the auxiliary results we are about to prove, resemble the case of a relative curve over a discrete valuation ring.

3.9. Let $R' = R(\sqrt{-1})$ be the algebraic closure of R , and let $B' = B[\sqrt{-1}]$, a valuation ring of R' that extends the valuation ring B of R . The maximal ideal of B' will be denoted \mathfrak{m}' , and we have $B'/\mathfrak{m}' = \mathbb{C}$. The valuation v on R (see 3.1), resp. w on $R(C)$ (see 3.2), extends uniquely to a valuation on R' , resp. on $R'(C)$, and we use the same letter v , resp. w , to denote this extension. The residue field of the valuation v on R' is \mathbb{C} , and the residue field of the valuation w on $R'(C)$ is $\mathbb{C}(C)$, the complex function field of the curve C . Given $g \in R'(C)$ with $w(g) \geq 0$, we denote the residue class of g in $\mathbb{C}(C)$ by \bar{g} . Also, we write $B'[C] = \mathbb{R}[C] \otimes B'$ and $R'[C] = \mathbb{R}[C] \otimes R'$. Again we have $B'[C] = \{f \in R'[C] : w(f) = 0\}$.

We consider the natural specialization map

$$C(B') \rightarrow C(\mathbb{C}), \quad \eta \mapsto \bar{\eta}$$

defined by composing homomorphisms $\mathbb{R}[C] \rightarrow B'$ with the residue map $B' \rightarrow B'/\mathfrak{m}' = \mathbb{C}$. Note that $\eta \in C(B')$ specializes to $\xi \in C(\mathbb{C})$ (that is, $\bar{\eta} = \xi$), if and only if $h(\eta) = 0$ implies $\bar{h}(\xi) = 0$, for every $h \in B[C]$. Given $\xi \in C(\mathbb{C})$, we'll use the notation

$$U(\xi) := \{\eta \in C(B') : \bar{\eta} = \xi\}.$$

The maximal ideal of $B[C]$ associated with ξ is denoted $M_\xi = \{f \in B[C] : \bar{f}(\xi) = 0\}$, see 3.3.

The zero or pole order of a rational function g on a nonsingular curve in a geometric point ξ will be denoted by $\text{ord}_\xi(g)$. Thus, given $f \in B'[C]$ and $\eta \in C(R')$, the symbol $\text{ord}_\eta(f)$ denotes the vanishing order of f in the point η of the generic fibre. For $\xi \in C(\mathbb{C})$, on the other hand, the symbol $\text{ord}_\xi(\bar{f})$ denotes the vanishing order in ξ of the restriction \bar{f} of f to the special fibre. Below (Proposition 3.13) we'll show how the vanishing orders of f in points of the generic fibre determine the vanishing orders of \bar{f} in the points of the special fibre.

Lemma 3.10. *Let $g \in R(C)^*$ satisfy $w(g) = 0$, let $\xi \in C(\mathbb{C})$ be a geometric point of the special fibre, and assume $\text{ord}_\eta(g) \geq 0$ for every $\eta \in U(\xi)$. Then there exist $0 \neq f, h \in B[C]$ with $g = \frac{f}{h}$ and $\bar{h}(\xi) \neq 0$.*

In other words, g lies in the localized ring $B[C]_{M_\xi}$.

Proof. We can write $g = \frac{b}{a}$ with $0 \neq a, b \in R[C]$, and clearly we can assume $w(a) = w(b) = 0$. This means that $a, b \in B[C]$.

Let η_1, \dots, η_r be the zeros of a in $U(\xi)$, and let ζ_1, \dots, ζ_s be the remaining zeros of a in $C(R')$. Since none of the ζ_j specializes to ξ , there exists $h \in B[C]$ satisfying $\bar{h}(\xi) \neq 0$ and $\text{ord}_{\zeta_j}(h) \geq \text{ord}_{\zeta_j}(a)$ for $j = 1, \dots, s$.

For any point $\eta \in C(R')$ we claim that $\text{ord}_\eta(bh) \geq \text{ord}_\eta(a)$ holds. Indeed, this is trivial if $a(\eta) \neq 0$. For $\eta \in \{\zeta_1, \dots, \zeta_s\}$ it is so by the choice of h . For $\eta \in \{\eta_1, \dots, \eta_r\}$ we have $\text{ord}_\eta(b) \geq \text{ord}_\eta(a)$ by the assumption on g . So $gh = \frac{bh}{a}$ lies in $R[C]$. Since $w(gh) = 0$, we have $gh \in B[C]$. \square

Lemma 3.11. *Let $f, g \in B'[x, y]$ be such that the coefficient-wise reduced polynomials $\bar{f}, \bar{g} \in \mathbb{C}[x, y]$ are not identically zero. Assume $f(0, 0) = g(0, 0) = 0$, and assume that the curves $\bar{f} = 0$ and $\bar{g} = 0$ in \mathbb{C}^2 intersect transversally at $(0, 0)$.*

Then the curves $f = 0$ and $g = 0$ in R'^2 intersect transversally at $(0,0)$, and they do not intersect in any point $(a,b) \neq (0,0)$ with $a, b \in \mathfrak{m}'$.

Proof. The gradient vectors of f and g at the origin lie in B'^2 , and by assumption they are linearly independent modulo \mathfrak{m}' . Hence they are linearly independent in R'^2 , which is the first assertion. After a linear change of coordinates we can assume

$$f = x + \sum_{d \geq 2} f_d(x, y), \quad g = y + \sum_{d \geq 2} g_d(x, y)$$

where $f_d, g_d \in B'[x, y]$ are homogenous polynomials of degree d , for $d \geq 2$. Let $(0,0) \neq (a,b) \in \mathfrak{m}' \times \mathfrak{m}'$, and assume $v(a) \leq v(b)$. Since $v(a) > 0$ we see that $v(f(a,b) - a) > v(a)$, whence $v(f(a,b)) = v(a)$, and therefore $f(a,b) \neq 0$. Likewise, $v(a) \geq v(b)$ implies $v(g(a,b)) = v(b)$ and $g(a,b) \neq 0$. \square

Lemma 3.12. *Let $\eta \in C(B')$, let $\xi = \bar{\eta} \in C(\mathbb{C})$.*

- (a) *There is $s \in B'[C]$ satisfying $s(\eta) = 0$ and $\text{ord}_\xi(\bar{s}) = 1$.*
- (b) *If $\eta \in C(B)$ then s can be found in $B[C]$.*
- (c) *Any s as in (a) satisfies $\text{ord}_\eta(s) = 1$ and $s(\eta') \neq 0$ for any $\eta' \in U(\xi) \setminus \{\eta\}$.*

Proof. Choose $t \in B'[C]$ such that $\bar{t} \in \mathbb{C}[C]$ is a local uniformizer at $\bar{\eta}$. Then $t(\eta) \in \mathfrak{m}'$. Putting $s := t - t(\eta)$ gives $s \in B'[C]$ with $s(\eta) = 0$ and with $\bar{s} = \bar{t}$, hence $\text{ord}_{\bar{\eta}}(\bar{s}) = 1$. If η is real, i.e. $\eta \in C(B)$, then t (and therefore s) can be found in $B[C]$. This proves (a) and (b).

(c) The question is local around the point $\bar{\eta} \in C(\mathbb{C})$. Zariski locally around any given point, any nonsingular curve is isomorphic to a Zariski open subset of a plane curve. Therefore we can assume that C is a (possibly singular) closed curve in $\mathbb{A}_{\mathbb{C}}^2$, and that $\xi = (0,0)$ is a nonsingular point of C . Now assertion (c) follows from Lemma 3.11. \square

Proposition 3.13. *Let $f \in B'[C]$ satisfy $w(f) = 0$. The vanishing order of \bar{f} in a point $\xi \in C(\mathbb{C})$ satisfies*

$$\text{ord}_\xi(\bar{f}) = \sum_{\eta \in U(\xi)} \text{ord}_\eta(f).$$

Proof. Let e denote the right hand sum in the assertion, and let

$$\{\eta \in U(\xi) : f(\eta) = 0\} =: \{\eta_1, \dots, \eta_r\}$$

(a finite set of points). For every $i = 1, \dots, r$, choose $s_i \in B'[C]$ with $w(s_i) = 0$, $s_i(\eta_i) = 0$ and $\text{ord}_\xi(\bar{s}_i) = 1$, according to Lemma 3.12(a). Moreover, put $e_i := \text{ord}_{\eta_i}(f)$. Let $s := s_1^{e_1} \cdots s_r^{e_r} \in B'[C]$, then we have $w(s) = 0$ and $\text{ord}_\xi(\bar{s}) = e_1 + \dots + e_r = e$. For any $\eta \in U(\xi)$ with $f(\eta) \neq 0$ we have $s(\eta) \neq 0$, by Lemma 3.12(c). Hence the rational function $g := \frac{f}{s} \in R'(C)^*$ has $\text{ord}_\eta(g) = 0$ for any $\eta \in C(B')$ with $\bar{\eta} = \xi$. Applying Lemma 3.10 to g and g^{-1} shows that g is a unit in the localized ring $B'[C]_{M_\xi}$. Thus $\bar{g}(\xi) \neq 0$, and therefore $\text{ord}_\xi(\bar{f}) = \text{ord}_\xi(\bar{s}) = e$. \square

4. MAIN THEOREM

The following fact is well-known:

Theorem 4.1. *Let C be a nonsingular affine curve over \mathbb{R} , and let $K \subseteq C(\mathbb{R})$ be a compact semi-algebraic set. Then the saturated preordering $\mathcal{P}(K)$ of K in $\mathbb{R}[C]$ is finitely generated.*

4.2. See [25] Theorem 5.21. More precisely, $\mathcal{P}(K)$ can be generated by two elements (even as a quadratic module), and can in fact be generated by a single element whenever K has no isolated points. If $K = C(\mathbb{R})$ (assumed to be compact) we have $\mathcal{P}(K) = \Sigma \mathbb{R}[C]^2$.

Let us briefly recall how this can be proved. When K has no isolated points, then $\mathcal{P}(K)$ is generated by any $f \in \mathcal{P}(K)$ which has simple zeros in the boundary points of K and has no other zeros in K (one can show that such f exists). This follows from the archimedean local-global principle (see Theorem 4.8 below). In the general case, let ξ_1, \dots, ξ_r be the isolated points of K . We modify the set K by replacing each isolated point ξ_i with a small closed interval $[\xi_i, \eta_i]$ on $C(\mathbb{R})$, for which $\eta_i \neq \xi_i$ lies on the same connected component of $C(\mathbb{R})$ as ξ_i , and the interval is so small that $[\xi_i, \eta_i] \cap K = \{\xi_i\}$. Let K_1 be the modified set obtained in this way, and note that K_1 has no isolated points. Let K_2 be a second such modification of K in which ξ_i gets replaced by $[\eta'_i, \xi_i]$, where $\eta'_i \neq \xi_i$ is again chosen close to ξ_i , but such that η_i and η'_i lie on opposite sides of ξ_i on the local branch of $C(\mathbb{R})$ around ξ_i . Then, by the first part of the argument, there exists a single generator f_j of $\mathcal{P}(K_j)$, for both $j = 1, 2$. Again using the archimedean local-global principle, one concludes that $\mathcal{P}(K)$ is generated by f_1 and f_2 .

The following theorem, resp. its corollary (to which the theorem is equivalent), is the first main result of this paper:

Theorem 4.3. *Let C be a nonsingular affine curve over \mathbb{R} , let $K \subseteq C(\mathbb{R})$ be a compact semi-algebraic set, and let $T = \mathcal{P}(K)$ be the saturated preordering of K in $\mathbb{R}[C]$. For any real closed field R containing \mathbb{R} , the preordering T_R generated by T in $R[C]$ is saturated as well.*

Corollary 4.4. *For C and K as in 4.3, the preordering $\mathcal{P}(K)$ in $\mathbb{R}[C]$ is stable.*

Proof. By [27] Corollary 3.8, $T = \mathcal{P}(K)$ is stable if and only if T_R is saturated in $R[C]$, for every real closed field R containing \mathbb{R} . So the assertion follows from Theorem 4.3. \square

Remarks 4.5.

1. Assume that the nonsingular affine curve C is irreducible. When C is rational, the assertions of Theorem 4.3 and Corollary 4.4 are true regardless whether K is compact or not. More precisely, assume that C is a nonsingular rational affine curve, and let $K \subseteq C(\mathbb{R})$ be any closed semi-algebraic subset. Then the saturated preordering $\mathcal{P}(K)$ of K in $\mathbb{R}[C]$ is finitely generated, and is stable. This is well-known, and essentially elementary.

2. When C has genus one and $C(\mathbb{R})$ is compact, 4.3 and 4.4 were proved for $K = C(\mathbb{R})$ in [30]. In all other cases of positive genus, these results are new.

3. When C has genus ≥ 1 and $K \subseteq C(\mathbb{R})$ is a closed semi-algebraic set that is not compact, two situations can occur. Either K is virtually compact (see 7.1 below); in this case we'll prove later that the above results remain true (Theorem 7.2 below). Or else K fails to be virtually compact; then it is known that the preordering $\mathcal{P}(K)$ fails to be finitely generated ([25] Theorem 5.21), and so the notion of stability does not even make sense for it.

4.6. We start giving the proof of Theorem 4.3. Let B be the convex hull of \mathbb{R} in R . We shall work in the ring $B[C] = \mathbb{R}[C] \otimes B$, and shall use the auxiliary results from Section 3. In particular, we use the notation introduced there. Let T_B be the preordering generated by T in $B[C]$. Theorem 4.3 asserts that T_R contains every $g \in R[C]$ with $g \geq 0$ on K_R . It suffices to show that T_B contains every $f \in B[C]$ with $f \geq 0$ on K_R and with $w(f) = 0$. Indeed, given $g \in R[C]$ with $g \geq 0$ on K_R , we find $0 \neq b \in R$ with $w(g) = v(b^2)$, and hence with $b^{-2}g \in B[C]$ and $w(b^{-2}g) = 0$. Knowing $b^{-2}g \in T_B$ clearly implies $g \in T_R$.

The following easy observation is crucial for the entire proof:

Lemma 4.7. *The preordering T_B in $B[C]$ is archimedean.*

Proof. According to Schmüdgen's theorem [31], the preordering T in $\mathbb{R}[C]$ is archimedean. Let $f \in B[C] = \mathbb{R}[C] \otimes B$. Since $T - T = \mathbb{R}[C]$, we can write f in the form $f = \sum_{i=1}^r f_i \otimes b_i$ with $f_i \in T$ and $b_i \in B$ ($i = 1, \dots, r$). Since T is archimedean, there exists $0 < c_1 \in \mathbb{R}$ with $c_1 - f_i \in T$ ($i = 1, \dots, r$). By the definition of B there exists $0 < c_2 \in R$ with $b_i \leq c_2$ in R for every i , and hence $c_2 - b_i$ is a square in B ($i = 1, \dots, r$). We conclude that

$$rc_1c_2 - f = c_2 \sum_{i=1}^r (c_1 - f_i) \otimes 1 + \sum_{i=1}^r f_i \otimes (c_2 - b_i)$$

lies in T_B . \square

Lemma 4.7 allows us to use the archimedean local-global principle, which we recall:

Theorem 4.8. ([26] Corollary 2.10) *Let A be a ring (containing $\frac{1}{2}$), let P be an archimedean preordering in A , and let f be an element of the saturation of P . Then f lies in P if (and only if) f lies in $P_{\mathfrak{m}}$ for every maximal ideal \mathfrak{m} of A .*

Here $P_{\mathfrak{m}}$ is the preordering generated by P in the localized ring $A_{\mathfrak{m}}$.

4.9. Resuming the proof of Theorem 4.3, fix $f \in B[C]$ with $w(f) = 0$ and with $f \geq 0$ on K_R . We have to show $f \in T_B$. From Lemma 3.5 we know that $f \geq 0$ on X_K , that is, f lies in the saturation of T_B in $B[C]$.

According to Lemma 4.7, we can apply Theorem 4.8 to T_B . Therefore it suffices to prove, for every maximal ideal M of $B[C]$, that f lies in T_M , the preordering generated by T in the local ring $B[C]_M$. Fix M , and let $X_{K,M} = X_K \cap \text{Sper } B[C]_M$. This is the basic closed constructible subset of $\text{Sper } B[C]_M$ associated with T_M .

If $f > 0$ on $X_{K,M}$, then $f \in T_M$ by [29], Proposition 2.1. So we can assume that there exists $\alpha \in X_K$ with $f \in \text{supp}(\alpha) \subseteq M$. The hypotheses of Lemma 3.7 apply to M , since $w(f) = 0$ implies $\text{supp}(\alpha) \not\subseteq \mathbb{R}[C] \otimes \mathfrak{m}$. By Lemma 3.7, therefore, we have $M = M_{\xi}$ for some point $\xi \in K$. From 3.9 recall the notation

$$U(\xi) = \{\eta \in C(B') : \bar{\eta} = \xi\}.$$

Lemma 4.10. *For every point $\eta \in U(\xi)$ with $f(\eta) = 0$, there exists an element $p_{\eta} \in T_B$ with $w(p_{\eta}) = 0$ such that $p_{\eta}(\eta) = 0$ and*

$$\text{ord}_{\xi}(\bar{p}_{\eta}) = \begin{cases} 1 & \text{if } \eta \in C(R), \eta \notin \text{int}(K_R), \\ 2 & \text{if } \eta \in C(R), \eta \in \text{int}(K_R), \\ 2 & \text{if } \eta \notin C(R). \end{cases}$$

Here $\text{int}(K_R)$ denotes the interior (relative to $C(R)$) of the semi-algebraic set $K_R \subseteq C(R)$.

4.11. Before giving the proof of Lemma 4.10, we show how the proof of Theorem 4.3 can be completed using this lemma. Let us decompose the set

$$\{\eta \in U(\xi) \cap C(R) : f(\eta) = 0\} = \{\eta_1, \dots, \eta_r\} \cup \{\zeta_1, \dots, \zeta_s\}$$

in such a way that the η_i lie in $\text{int}(K_R)$ and the ζ_j don't. Note that f has even order in any of the points η_i . Among the nonreal zeros of f in $U(\xi)$, choose a subset $\{\omega_1, \dots, \omega_t\}$ that contains exactly one representative from each pair of complex conjugate points. Then put

$$p := \prod_{i=1}^r (p_{\eta_i})^{\frac{1}{2} \text{ord}_{\eta_i}(f)} \cdot \prod_{j=1}^s (p_{\zeta_j})^{\text{ord}_{\zeta_j}(f)} \cdot \prod_{k=1}^t (p_{\omega_k})^{\text{ord}_{\omega_k}(f)},$$

with the p_η chosen as in Lemma 4.10. Then $p \in T_B$. By Proposition 3.13 we have

$$\text{ord}_\xi(\bar{f}) = \sum_{i=1}^r \text{ord}_{\eta_i}(f) + \sum_{j=1}^s \text{ord}_{\zeta_j}(f) + 2 \sum_{k=1}^t \text{ord}_{\omega_k}(f),$$

and this number is also equal to $\text{ord}_\xi(\bar{p})$. It follows that $g := \frac{f}{p}$ is a unit in the local ring $B[C]_M$.

We would like g to take nonnegative values in all real points $\eta \in U(\xi) \cap K_R$. This obviously is the case whenever $p(\eta) \neq 0$. By continuity, it is also true whenever η is not an isolated point of K_R . The case when η is an isolated point of K_R can occur only for $\eta = \xi \in C(\mathbb{R})$, and when ξ is an isolated point of K . This case can be dealt with by modifying one of the local factors p_ξ in the product p . We will explain this in the course of the proof of Lemma 4.10, see 4.12.

Once this point is settled, the unit g of $B[C]_M$ takes positive values on the set $X_{K,M}$ associated with the preordering T_M , using Lemma 3.5. Hence, by another application of [29] Proposition 2.1, we conclude that g lies in T_M . As a consequence we get $f = pg \in T_M$, as desired.

4.12. For the proof of Lemma 4.10 we need to distinguish several cases. First assume $\eta \notin C(R)$. By Lemma 3.12 there exists $s \in B'[C]$ with $s(\eta) = 0$ and $\text{ord}_\xi(\bar{s}) = 1$. Let τ be the automorphism of $R'[C]$ of order two that is induced by complex conjugation. Then $p_\eta := s \cdot \tau(s)$ is a sum of two squares in $B[C]$, and clearly satisfies $p_\eta(\eta) = 0$ and $\text{ord}_\xi(\bar{p}_\eta) = 2$.

When η is an interior point of K_R , choose $s \in B[C]$ with $s(\eta) = 0$ and $\text{ord}_\xi(\bar{s}) = 1$, see Lemma 3.12(b). Then $p_\eta := s^2$ will do the job.

Now assume that η is real but not an interior point of K_R . Then necessarily ξ is a boundary point of K . Since T is saturated, there exists $t \in T$ with $\text{ord}_\xi(t) = 1$. As a function on $C(R)$, t therefore changes sign in ξ and has no other zero in $U(\xi)$. Since $\eta \in U(\xi)$ is not an interior point of K_R , we have $t(\eta) \leq 0$. So $p_\eta := t - t(\eta)$ lies in T_B and has the desired properties. With this the lemma is proved, but it remains to add a remark fixing the isolated point case, see 4.11. So assume that ξ is an isolated point of K . In this case, since T is saturated, there is a second element $t' \in T$ with $\text{ord}_\xi(t') = 1$ such that $tt' \leq 0$ in a neighborhood of ξ (on K , or on K_R). In the definition of p in 4.11, we replace (in the case when $f(\xi) = 0$) one of the local factors t at ξ by t' . That is, we replace $t^{\text{ord}_\xi(f)}$ by $t^{\text{ord}_\xi(f)-1}t'$. In this way we have changed the sign of $g(\xi)$ in 4.11. We see that we can ensure that we always have $g(\xi) > 0$.

This proves Lemma 4.10, and this completes the proof of Theorem 4.3. \square

5. SEMIDEFINITE REPRESENTATIONS IN THE COMPACT CASE

Now we use moment relaxation to obtain semidefinite representations from the results of the previous section.

5.1. Let $K \subseteq \mathbb{R}^n$ be a compact and convex semi-algebraic set whose set $\text{Ex}(K)$ of extremal points has (semi-algebraic) dimension at most one. The closure $K_0 = \overline{\text{Ex}(K)}$ is a compact semi-algebraic set satisfying $\dim(K_0) \leq 1$, and $K = \text{conv}(K_0)$ by the Krein-Milman theorem. We assume that no nonzero linear polynomial vanishes identically on K .

Let $C_0 \subseteq \mathbb{A}^n$ be the (reduced) Zariski closure of K_0 in affine n -space, so C_0 is an \mathbb{R} -variety (usually reducible) of dimension ≤ 1 . Let $C'_0 \rightarrow C_0$ be the normalization of C_0 . On \mathbb{R} -points, the map $C'_0(\mathbb{R}) \rightarrow C_0(\mathbb{R})$ may fail to be surjective. Namely, when ξ is an isolated point of $C_0(\mathbb{R})$ that is singular as a point on C_0 , then ξ is not in the image of $C'_0(\mathbb{R})$. Let ξ_1, \dots, ξ_k be the isolated points of $C_0(\mathbb{R})$ that lie in K_0

and are singular points of C_0 , and write $P_i = \text{Spec}(\mathbb{R})$ for $i = 1, \dots, k$. Finally let

$$C_1 = C'_0 \amalg P_1 \amalg \dots \amalg P_k$$

(disjoint sum), and let $\phi: C_1 \rightarrow C_0 \subseteq \mathbb{A}^n$ be the canonical morphism sending P_i to ξ_i ($i = 1, \dots, k$). Let K_1 be the preimage of K_0 in $C_1(\mathbb{R})$. Then K_1 is a compact semi-algebraic subset of $C_1(\mathbb{R})$, and we have $\phi(K_1) = K_0$.

If C_0 has the irreducible components X_1, \dots, X_l , if we write $A_i = \mathbb{R}[X_i]$ and denote by A'_i the integral closure of A_i in its quotient field, the coordinate ring of C_1 is therefore

$$\mathbb{R}[C_1] = A'_1 \times \dots \times A'_l \times \mathbb{R} \times \dots \times \mathbb{R}$$

(with k factors \mathbb{R}). The saturated preordering $\mathcal{P}(K_1)$ of K_1 in $\mathbb{R}[C_1]$ is finitely generated (see Theorem 4.1) and stable (Theorem 4.4).

The morphism $C_1 \rightarrow C_0 \subseteq \mathbb{A}^n$ induces a homomorphism $\mathbb{R}[x_1, \dots, x_n] \rightarrow \mathbb{R}[C_1]$ of \mathbb{R} -algebras, which is injective when restricted to $L := \text{span}(1, x_1, \dots, x_n)$, by our assumption on K . We consider L as a linear subspace of $\mathbb{R}[C_1]$. Let $\Sigma = \Sigma\mathbb{R}[C_1]^2$, and choose $1 = h_0, h_1, \dots, h_r \in \mathbb{R}[C_1]$ with $\mathcal{P}(K_1) = \Sigma h_0 + \dots + \Sigma h_r$. Since $\mathcal{P}(K_1)$ is stable, there exists a tuple $W = (W_0, \dots, W_r)$ of finite-dimensional subspaces $W_i \subseteq \mathbb{R}[C_1]$ such that $L \cap \mathcal{P}(K_1)$ is contained in

$$M_W = \Sigma_{W_0} + \Sigma_{W_1} h_1 + \dots + \Sigma_{W_r} h_r$$

(c.f. 2.3). By Theorem 2.4 this implies that we have found a semidefinite representation for $\text{conv } \phi(K_1) = K$. We have thus proved:

Theorem 5.2. *Every compact convex semi-algebraic set $K \subseteq \mathbb{R}^n$ with $\dim \text{Ex}(K) \leq 1$ is a projected spectrahedron. A semidefinite representation of K can be obtained from a suitable moment relaxation.* \square

Example 5.3. To illustrate the construction in 5.1, let us consider the (rational) affine curve C_0 with equation

$$y^2 + x^2(x-1)(x-2) = 0.$$

The set $C_0(\mathbb{R})$ is compact and has the origin as an isolated point. To construct a semidefinite representation for the convex hull K of $C_0(\mathbb{R})$, we work in $A_1 = A'_0 \times \mathbb{R}$, where A'_0 is the integral closure of $A_0 = \mathbb{R}[C_0]$, i.e., $A'_0 = \mathbb{R}[x, z]/(z^2 + (x-1)(x-2))$ (where $y = xz$). Using the elements $1 = (1, 1)$, $u = (x, 0)$, $v = (z, 0)$ and $e = (1, 0)$ of A_1 , we let $L = \text{span}(1, u, uv)$, $W = \text{span}(1, e, u, v)$ and $U = WW = \text{span}(1, e, u, v, u^2, uv)$. The relaxation for K obtained from this data is exact. Using the basis $1 - e, e, u, v$ for W , we get K as the set of all $(\xi, \eta) \in \mathbb{R}^2$ for which there exist $a, b, c \in \mathbb{R}$ with

$$\begin{pmatrix} 1-c & 0 & 0 & 0 \\ 0 & c & \xi & a \\ 0 & \xi & b & \eta \\ 0 & a & \eta & 3\xi - b - 2c \end{pmatrix} \succeq 0.$$

Remark 5.4. Here are a few remarks on possible extensions of Corollary 4.4 and Theorem 5.2 to dimensions larger than one. Corollary 4.4 says, for K a compact semi-algebraic set on a nonsingular affine curve, that the saturated preordering $\mathcal{P}(K)$ is (finitely generated and) stable. Unfortunately, according to the main result of [27], there does not exist any semi-algebraic set K with $\dim(K) > 1$ to which this result would extend.

On the other hand, note that the application of moment relaxation to semidefinite representations of convex hulls requires much less than saturatedness and stability of some finitely generated preordering. All that is needed is partial (relative) versions of these conditions with respect to the space L of linear polynomials, see

Corollary 2.5. These may still be impossible to satisfy, but there is more freedom: Given a semi-algebraic set $K \subseteq \mathbb{R}^n$, one could hope to find a morphism $\phi: V \rightarrow \mathbb{A}^n$ of affine varieties and a basic closed set $K' \subseteq V(\mathbb{R})$ satisfying $\phi(K') = K$ such that the exactness conditions for the moment relaxation can be satisfied for K' and for the linear subspace in $\mathbb{R}[V]$ which is the image of L under ϕ^* . In 5.1 above, we have used this approach, replacing a singular curve by its normalization. In dimension greater than one, we do not know how far one can expect to get with such a strategy.

6. SEMIDEFINITE REPRESENTATIONS IN THE GENERAL CASE

Using the compact case, we now establish semidefinite representations for the closed convex hull of an arbitrary one-dimensional semi-algebraic set, and will deduce the dimension two case of the Helton-Nie conjecture. I am indebted to Tim Netzer who showed me how to obtain semidefinite representations for noncompact closed convex sets from such representations for compact sets.

Theorem 6.1. *Let $K \subseteq \mathbb{R}^n$ be the closed convex hull of a semi-algebraic set of dimension ≤ 1 . Then K has a semidefinite representation.*

6.2. Before we start the proof, we need to recall a few notions on convex sets and cones, for which we refer to [20]. Given a nonempty closed convex set $K \subseteq \mathbb{R}^n$, the *recession cone* of K is

$$\text{rc}(K) = \{x \in \mathbb{R}^n : K + x \subseteq K\},$$

and is a closed convex cone. Note that $\text{rc}(K)$ can also be described as the set of all existing limits $\lim_{\nu \rightarrow \infty} a_\nu x_\nu$ in \mathbb{R}^n , where x_ν is a sequence in K and a_ν is a null sequence of positive real numbers. The homogenization K^h of K is the closure of the convex cone $K' = \{(t, tx) : t \geq 0, x \in K\}$ in \mathbb{R}^{n+1} , and

$$K^h = K' \cup \{(0, y) : y \in \text{rc}(K)\}.$$

The original set K can clearly be recovered from its homogenization as $K = \{x \in \mathbb{R}^n : (1, x) \in K^h\}$. The extremal rays of K^h are the rays spanned by points $(1, x)$ with $x \in \text{Ex}(K)$, together with the rays spanned by points $(0, y)$ where $\mathbb{R}_+ y$ is an extremal ray of $\text{rc}(K)$.

6.3. Let $S \subseteq \mathbb{R}^n$ be a semi-algebraic set. A ray $\mathbb{R}_+ u$ (with $0 \neq u \in \mathbb{R}^n$) will be called an *asymptotic direction of S at infinity* if there exist continuous semi-algebraic paths $a(t)$ in \mathbb{R}_+ and $x(t)$ in S (with $0 < t \leq 1$) such that $a(t) \rightarrow 0$ and $a(t)x(t) \rightarrow u$ for $t \rightarrow 0$.

Proposition 6.4. *Let $S \subseteq \mathbb{R}^n$ be a nonempty closed semi-algebraic set, and let $K = \text{conv}(S)$ be its closed convex hull.*

- (a) *Each extremal point of K is contained in S .*
- (b) *Each extremal ray of $\text{rc}(K)$ is an asymptotic direction of S at infinity.*

Proof. For the proof of both parts we can assume $\text{rc}(K) \cap (-\text{rc}(K)) = \{0\}$. To prove (a) we are going to show that, whenever $\xi \in K \setminus \text{conv}(S)$, there exists $0 \neq u \in \text{rc}(K)$ with $\xi - u \in \text{conv}(S)$ (and hence ξ is not an extremal point of K). Given $\xi \in K$, there exist continuous semi-algebraic paths $a_i(t)$ in $[0, 1]$ and $x_i(t)$ in S , for $i = 0, \dots, n$ and $0 < t \leq 1$, such that $\sum_{i=0}^n a_i(t) \equiv 1$, and such that

$$x(t) = \sum_{i=0}^n a_i(t)x_i(t) \tag{3}$$

converges to ξ for $t \rightarrow 0$. Note that the limit $\alpha_i = \lim_{t \rightarrow 0} a_i(t)$ exists in $[0, 1]$ for every $0 \leq i \leq n$, and that $\sum_i \alpha_i = 1$.

We claim that the curves $a_i(t)x_i(t)$ ($i = 0, \dots, n$) are bounded for $t \rightarrow 0$. Indeed, assume that $a_i(t)x_i(t)$ is unbounded for at least one index i . Let $q > 0$ be the minimal rational number such that, for every $0 \leq i \leq n$, the curve $t^{-q}a_i(t)x_i(t)$ is bounded and therefore the limit $u_i := \lim_{t \rightarrow 0} t^{-q}a_i(t)x_i(t)$ exists in \mathbb{R}^n . For each index i we have $u_i \in \text{rc}(K)$, and $u_i \neq 0$ for at least one index i . Dividing (3) by t^q gives $\sum_{i=0}^n u_i = 0$, contradicting $\text{rc}(K) \cap (-\text{rc}(K)) = \{0\}$.

Since the curves $a_i(t)x_i(t)$ are all bounded, the limits $u_i := \lim_{t \rightarrow 0} a_i(t)x_i(t)$ exist in \mathbb{R}^n . If $x_i(t)$ is unbounded then $\alpha_i = 0$ and $u_i \in \text{rc}(K)$. If $x_i(t)$ is bounded then $\xi_i = \lim_{t \rightarrow 0} x_i(t)$ exists in S , and $u_i = \alpha_i \xi_i$. Let y denote the sum of the u_i for those indices i for which $x_i(t)$ is bounded, and let u be the sum of the remaining u_i . Then $y \in \text{conv}(S)$, $u \in \text{rc}(K)$ and $\xi = y + u$. This proves our assertion.

The proof of (b) is similar to the proof of (a). After making a translation we can assume $0 \in K$. Let $0 \neq u \in \text{rc}(K)$. Similar to (3) we have

$$\frac{1}{t}u - w(t) = \sum_{i=0}^n a_i(t)x_i(t)$$

($0 < t \leq 1$) with semi-algebraic paths $a_i(t)$ in $[0, 1]$ and $x_i(t)$ in S , where $\sum_i a_i(t) \equiv 0$ and $w(t)$ is a correction term with $|w(t)| < 1$. Multiplication with t gives

$$u - tw(t) = \sum_{i=0}^n ta_i(t)x_i(t).$$

For $t \rightarrow 0$, the summands on the right remain bounded, as shown above. Therefore the limit $u_i = \lim_{t \rightarrow 0} ta_i(t)x_i(t)$ exists in \mathbb{R}^n for $i = 0, \dots, n$, and $\mathbb{R}_+ u_i$ is an asymptotic direction of S at infinity (see 6.3) if $u_i \neq 0$. From $u = \sum_i u_i$ we see that if $\mathbb{R}_+ u$ is an extremal ray of $\text{rc}(K)$, then $\mathbb{R}_+ u = \mathbb{R}_+ u_i$ for some i , which proves (b). \square

6.5. We now give the proof of Theorem 6.1. Let $S \subseteq \mathbb{R}^n$ be a nonempty semi-algebraic set of dimension at most one, and let $K = \overline{\text{conv}(S)}$ be its closed convex hull. In order to prove that K has a semidefinite representation we may assume $\text{rc}(K) \cap (-\text{rc}(K)) = \{0\}$. For the homogenization $K^h \subseteq \mathbb{R}^{n+1}$ of K (see 6.2) this implies $K^h \cap (-K^h) = \{0\}$. So the dual cone $(K^h)^*$ of K^h in \mathbb{R}^{n+1} is full-dimensional, and we can pick an interior point w of $(K^h)^*$. The convex set

$$K_1 := \{x \in K^h : \langle x, w \rangle = 1\}$$

is compact, and K^h is (isomorphic to) the homogenization of K_1 .

The extremal rays of the convex cone K^h correspond to the extremal points of K and to the extremal rays of $\text{rc}(K)$, see 6.2. By Proposition 6.4, $\text{Ex}(K) \subseteq \overline{S}$ has dimension ≤ 1 , and $\text{rc}(K)$ has only finitely many extremal rays. Considering the set of extremal rays of K^h as a subset of the unit sphere in \mathbb{R}^{n+1} , this set therefore has dimension ≤ 1 . It follows that the set $\text{Ex}(K_1)$ of extremal points of K_1 has dimension ≤ 1 as well. So we can apply Theorem 5.2 to K_1 , and conclude that K_1 has a semidefinite representation. By Lemma 6.6 below, this implies that the cone $(K_1)^h \cong K^h$ has a semidefinite representation as well. This completes the proof of Theorem 6.1, since K , being an affine-linear section of K^h , also has a semidefinite representation.

Lemma 6.6. *Let $K \subseteq \mathbb{R}^n$ be a compact convex semi-algebraic set. If K has a semidefinite representation, then the homogenization K^h of K has a semidefinite representation as well.*

Proof. Since K is compact, $K^h = \{(t, tx) : t \geq 0, x \in K\}$ is simply the cone over K . Therefore, a semidefinite representation of K^h is obtained by homogenizing a

representation of K . In more detail, assume that

$$K = \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^k \ M(1, x) + N(y) \succeq 0\}$$

where $M(t, x) = tM_0 + \sum_{i=1}^n x_i M_i$ and $N(y) = \sum_{j=1}^k y_j N_j$ are linear systems of symmetric matrices. Homogenizing the linear matrix inequality we obtain a semidefinitely representable convex cone

$$C := \{(t, x) \in \mathbb{R}^{n+1} : t \geq 0, \exists y \in \mathbb{R}^k \ M(t, x) + N(y) \succeq 0\}$$

in \mathbb{R}^{n+1} . We claim that the obvious inclusion $K^h \subseteq C$ is an equality. Indeed, both convex cones are contained in the closed half-space $t \geq 0$, and they intersect the open half-space $t > 0$ in the same (nonempty) set. Since K^h is closed, this implies the reverse inclusion $C \subseteq K^h$. \square

Now we combine Theorem 6.1 with results of Netzer to show:

Theorem 6.7. (Helton-Nie conjecture in dimension two) *Every convex semi-algebraic subset of \mathbb{R}^2 is a projected spectrahedron.*

Proof. Let $K \subseteq \mathbb{R}^2$ be a convex semi-algebraic set. To prove that K has a semidefinite representation, we first consider the case when K is closed. If K contains a line, the assertion is obvious by reduction to a (closed) convex subset of \mathbb{R} . So we assume $\text{rc}(K) \cap (-\text{rc}(K)) = \{0\}$. Then $K = \text{conv Ex}(K) + \text{rc}(K) = \overline{\text{conv Ex}(K)} + \text{rc}(K)$ ([20] Theorem 18.5), and the set $\text{Ex}(K)$ is semi-algebraic of dimension ≤ 1 . By Theorem 6.1, $\overline{\text{conv Ex}(K)}$ is a projected spectrahedron. Since $\text{rc}(K)$ is clearly a projected spectrahedron, we see that K is one as well.

Now let $K \subseteq \mathbb{R}^2$ be an arbitrary convex semi-algebraic set. We can assume that K has nonempty interior. Let M be the set of points in the boundary $\partial K = \partial \overline{K}$ that do not lie in K . Then M is a semi-algebraic set with $\dim(M) \leq 1$, and we can decompose M set-theoretically as follows. Let M_0 be the relative interior of $M \cap \text{Ex}(\overline{K})$ inside $\partial \overline{K}$, and let \mathcal{F} be the (finite) set of one-dimensional faces of \overline{K} . For each $F \in \mathcal{F}$, let $M_F = F \cap M$. Moreover, let H_F be the open halfplane with $H_F \cap K \neq \emptyset$ whose boundary line contains F , and let $K_F = H_F \cup (F \cap K) = H_F \cup (\overline{H_F} \cap K)$. Then M is the union of M_0 with finitely many extremal points of \overline{K} and with $\bigcup_{F \in \mathcal{F}} M_F$. Accordingly, K is the intersection of $K_0 := \overline{K} \setminus M_0$ with finitely many sets $K_\xi := \overline{K} \setminus \{\xi\}$ (where $\xi \in \text{Ex}(\overline{K})$) and with the sets K_F ($F \in \mathcal{F}$).

Since a finite intersection of projected spectrahedra is again a projected spectrahedron, it suffices to show that each of K_0 , K_ξ and K_F as above is a projected spectrahedron. Each of the sets K_F is a union of an open halfplane H with a convex subset of the line ∂H . Using the result of [14], such K_F is a projected spectrahedron. (Due to the elementary nature of this situation, one can easily find an explicit semidefinite representation directly.) The sets K_ξ ($\xi \in \text{Ex}(\overline{K})$) are projected spectrahedra by [12] Proposition 3.1. For K_0 we use Netzer's construction from [12]. Let $N = \partial \overline{K} \setminus M_0$, a closed subset of $\partial \overline{K}$ with $K_0 = \text{int}(K) \cup N$, and let $T = \overline{\text{conv}(N)}$. Then T is a closed convex subset of \overline{K} , and is a projected spectrahedron by Theorem 6.1. By construction, and by Proposition 6.4, $T \cap \partial \overline{K} = N = \partial \overline{K} \setminus M_0$. Using the notation introduced in [12], let $(T \leftrightarrow \overline{K})$ denote the union of the relative interiors of all the faces of \overline{K} that meet T . We see that $(T \leftrightarrow \overline{K}) = \text{int}(K) \cup N = K_0$. By [12] Theorem 3.8, $(T \leftrightarrow \overline{K})$ is a projected spectrahedron, which proves our theorem. \square

7. STABILITY IN THE VIRTUALLY COMPACT CASE

7.1. Let C be an irreducible affine curve over \mathbb{R} , and let $K \subseteq C(\mathbb{R})$ be a closed semi-algebraic subset. Adopting the terminology of [25], [28], we say that K is

virtually compact if there exists an irreducible affine curve C_1 containing C as a Zariski open subset, in such a way that the points in $C_1 \setminus C$ are nonsingular on C_1 and the closure K_1 of K in $C_1(\mathbb{R})$ is compact. Equivalently, K is virtually compact if and only if there exists a nonconstant regular function $f \in \mathbb{R}[C]$ that is bounded on K .

When the affine curve C is not necessarily irreducible, a closed semi-algebraic set $K \subseteq C(\mathbb{R})$ is called *virtually compact* if $K \cap C'(\mathbb{R})$ is virtually compact on C' , for every irreducible component C' of C . A closed semi-algebraic set $K \subseteq \mathbb{R}^n$ of dimension ≤ 1 is called *virtually compact* if it has this property with respect to its Zariski closure C .

We show that the analogues of the stability results from Section 4 remain true for virtually compact sets K :

Theorem 7.2. *Let C be an irreducible nonsingular affine curve over \mathbb{R} , and let $K \subseteq C(\mathbb{R})$ be a closed semi-algebraic subset that is virtually compact. Then the saturated preordering $\mathcal{P}(K)$ in $\mathbb{R}[C]$ is finitely generated and stable.*

Proof. That $\mathcal{P}(K)$ is finitely generated was already proved (in greater generality) in [25] Theorem 5.21. We are going to reprove this fact here using a different reasoning, because we'll need the same argument to prove stability. Since the theorem has already been proved when K is compact, we can assume that K is not compact. In particular, the set K is infinite.

Let C_1 and K_1 be as in 7.1. Then $\mathbb{R}[C_1]$ is a subring of $\mathbb{R}[C]$. Let $T = \mathcal{P}_C(K)$ (the saturated preordering of K in $\mathbb{R}[C]$), and let $T_1 = \mathcal{P}_{C_1}(K_1)$ (the saturated preordering of K_1 in $\mathbb{R}[C_1]$). Since K_1 is compact, the preordering T_1 in $\mathbb{R}[C_1]$ is finitely generated, according to Theorem 4.1. So there are nonzero elements $1 = h_0, h_1, \dots, h_r \in \mathbb{R}[C_1]$ such that T_1 is the quadratic module in $\mathbb{R}[C_1]$ generated by h_1, \dots, h_r . (We can even do with $r \leq 2$, see 4.2.) We'll prove that $T = \mathcal{P}_C(K)$ is generated by h_1, \dots, h_r in $\mathbb{R}[C]$.

Let \tilde{C} be the nonsingular projective curve over \mathbb{R} that contains C_1 as an open dense subscheme. We consider Weil divisors on \tilde{C} , and regard them as conjugation-invariant Weil divisors on the complexified curve $\tilde{C}_{\mathbb{C}}$. Since $\tilde{C}(\mathbb{R}) \neq \emptyset$, we have $\text{Pic}(\tilde{C}) = \text{Pic}(\tilde{C}_{\mathbb{C}})^{\text{Gal}(\mathbb{C}/\mathbb{R})}$. Let J be the Jacobian variety of \tilde{C} , an abelian variety over \mathbb{R} .

Let $C_1(\mathbb{C}) \setminus C(\mathbb{C}) = \{Q_1, \dots, Q_s\}$, and let $0 \neq f \in \mathbb{R}[C]$ with $f|_K \geq 0$. For $i = 1, \dots, s$ let $m_i \geq 0$ be an integer satisfying $2m_i + \text{ord}_{Q_i}(f) \geq 0$, and consider the divisor

$$D = \sum_{i=1}^s m_i Q_i$$

on \tilde{C} . Choose a point $Q \in \tilde{C}(\mathbb{C}) \setminus C_1(\mathbb{C})$, and let $E = Q + \bar{Q}$ (again a divisor on \tilde{C} , the case $Q = \bar{Q}$ is allowed, bar denoting complex conjugation). There exist integers $l, n \geq 1$ such that the divisor $nE - lD$ has degree zero, and such that the divisor class $[nE - lD] \in J(\mathbb{R})$ lies in the identity connected component $J(\mathbb{R})_0$ of the compact real Lie group $J(\mathbb{R})$. Fix an arbitrary \mathbb{R} -point P_0 in the interior $\text{int}(K)$ of K relative to $C(\mathbb{R})$. By the argument in [23], 2.11 and 2.12, there is an integer $k \geq 1$ such that, for every $\alpha \in J(\mathbb{R})_0$, there exist $2k$ points $P_1, \dots, P_{2k} \in \text{int}(K)$ with

$$\alpha = \sum_{j=1}^{2k} [P_j - P_0].$$

Applying this to the divisor class $\alpha := [nE - lD - k(2P_0 - E)]$ (which lies in $J(\mathbb{R})_0$, c.f. [23] Lemma 2.6), we conclude that there exist $P_1, \dots, P_{2k} \in \text{int}(K)$ such that

$$lD + \sum_{j=1}^{2k} P_j \sim (n+k)E$$

on \tilde{C} . Since $\text{supp}(E)$ is disjoint to C_1 , there exists $0 \neq h \in \mathbb{R}[C_1]$ such that the divisor of h on C_1 is $lD + \sum_{j=1}^{2k} P_j$. Since $\text{ord}_{Q_i}(h^2 f) \geq 2lm_i + \text{ord}_{Q_i}(f) \geq 0$, we see that $h^2 f$ lies in $\mathbb{R}[C_1]$. Moreover, every zero of h on C is real and is an interior point of K . In addition, we can ensure that h has no common zero with any of h_0, \dots, h_r .

Since $f \geq 0$ on K , and since K is dense in K_1 , it follows that $h^2 f \geq 0$ on K_1 . So $h^2 f \in T_1$, which means that there is an identity

$$h^2 f = \sum_{i=0}^r \sum_j p_{ij}^2 h_i$$

with suitable $p_{ij} \in \mathbb{R}[C_1]$. Since any zero of h on C is real and is an interior point of K , it follows that each summand $p_{ij}^2 h_i$ of the right hand sum is divisible (inside $\mathbb{R}[C]$) by h^2 , see [23] Lemma 0.1. By the choice of h , none of the h_i vanishes in any of the zeros of h . Hence we even have $h \mid p_{ij}$ (inside $\mathbb{R}[C]$), for all indices i, j . Dividing we conclude that f lies in the quadratic module generated by h_0, \dots, h_r in $\mathbb{R}[C]$.

We have thus proved that $T = \mathcal{P}_C(K)$ is finitely generated in $\mathbb{R}[C]$. To prove that T is stable is equivalent to proving the following assertion (c.f. [27] Corollary 3.8): Let R be any real closed extension field of \mathbb{R} . Then the preordering T_R generated by T in $R[C]$ is saturated.

To prove this, let $0 \neq f \in R[C]$ be nonnegative on K_R , where K_R denotes the extension of the semi-algebraic set $K \subseteq C(\mathbb{R})$ to a semi-algebraic subset of $C(R)$. Arguing literally as in the first part of the proof, we find $h \in \mathbb{R}[C_1]$ such that $h^2 f \in R[C_1]$, and such that any zero of h on C is real and is an interior point of K in which $h_0 \cdots h_r$ does not vanish. Completing the argument exactly as before, we see that f lies in the quadratic module of $R[C]$ generated by h_0, \dots, h_r . In other words, $f \in T_R$, as desired. The theorem is proved. \square

Remarks 7.3.

1. The proof has shown that $T = \mathcal{P}_C(K)$ is the preordering generated by the preordering $T_1 = \mathcal{P}_{C_1}(K_1)$ of $\mathbb{R}[C_1]$ in the larger ring $\mathbb{R}[C]$. The same fact is readily deduced from [25] Theorem 5.5.

2. Without using Theorem 6.1, we can conclude from Theorem 7.2: For any closed semi-algebraic set $K \subseteq \mathbb{R}^n$ of dimension ≤ 1 which is virtually compact, the closed convex hull $\overline{\text{conv}(K)}$ is the closure of a projected spectrahedron. Indeed, by arguing exactly as in 5.1 one finds a projected spectrahedron sandwiched between $\text{conv}(K)$ and $\overline{\text{conv}(K)}$. Note however that this conclusion is weaker than Theorem 6.1 in several respects.

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